On an Stabilizable Control Force for the Inverted Pendulum Cart System

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Abstract. A novel control force is presented to stabilize the inverted pendulum mounted on a cart. The control force is based on proposing a partial feedback linearization, in conjunction with Lyapunov's second method. This simple, but efficient strategy guarantees that the closed-loop system is locally asymptotically stable around the unstable equilibrium point. Additionally, the controller has a very large attraction domain and it is robust with respect to damping .

Keywords: Classical Mechanics; Nonlinear Systems, Lyapunov's second method;

1 Introduction

The study of the simple inverted pendulum cart system has been one of the most interesting problems in classical mechanics and modern control theory. The device consists of a pole whose pivot point is mounted on a cart. The pendulum is free to rotate about its pivot point. The cart can move horizontally perpendicular to the axis's pendulum and is actuated by a horizontal force. The mechanical problem is to bring the pendulum from large initial pendulum deviation to the upper unstable equilibrium position by moving the cart on the horizontal plane. This system has attracted the attention of many researchers, as seen by a growing list of articles (for example, see [1], [2], [3], [4], [5], [6] and [7]). The interest is due to the fact that, the device is non-feedback linearizable by means of dynamics state feedback (see [8]), and hence, it is not linearizable by means of dynamic state feedback control either. This obstacle makes it especially difficult to perform some controlled maneuvers; for instance, there is no continuous force which globally stabilizes the upright equilibrium of the pendulum with zero displacement of the cart [9]. Nevertheless, the problem can be solved producing at least one discontinuity in the acceleration cart. However, it is well-known how to construct a linear locally stabilizing controller [10] but, the linear based controls design presents the inconvenience of having a very small domain of attraction.

In this article, we present a control force to locally asymptotically stabilize the inverted pendulum cart system around its unstable equilibrium point[1] , for a very large attraction domain.
Also, intuitively the proposed control force allows us to transfer the pendulum from the stable equilibrium point to the unstable equilibrium points, which are the lower and the upper resting angular position, respectively. This task is done by means of switching a suitable parameter. The system stability is demonstrated by using Lyapunov's second method.
This paper is organized as follows. Section 1 introduces the normalized non-linear model of mechanical device. Next, a partial feedback linearization of the non-linear equations is obtained. Section 2 presents a non-linear controller for the stabilization of the device, and some computer simulation results depicting the performance of the closed-loop system are presented. Finally, Section 3 gives the conclusions.

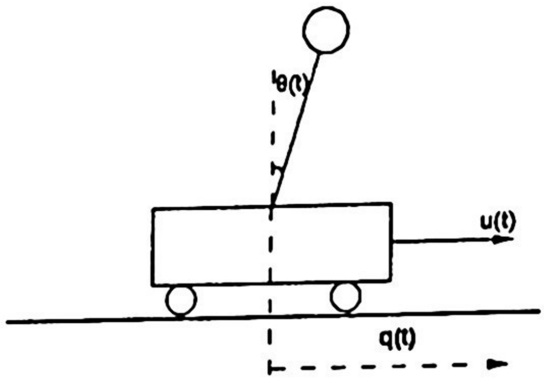


Fig. 1. The inverted pendulum cart system

2 The Inverted Pendulum Cart System

Consider the traditional inverted pendulum mounted on a cart (see Figure 1). The nonlinear model of the system, which can be obtained from either the Newton or the Euler-Lagrage equations, see more detail (Lozano and Fantoni) is given by

$$mL\cos\theta\,\ddot{x} + mL^2\ddot{\theta} - gmL\sin\theta = 0$$
$$(\mathrm{M}+\mathrm{m})\ddot{x} + Lm\cos\theta\ddot{\theta} - mL\dot{\theta}^2\sin\theta = f$$

where x is the cart displacement, θ is the angle that the pendulum forms with the vertical, f is the force applied to the cart, acting as the control input. M and

[1] Which is the upper angular position with zero displacement of the cart

m stand for the cart mass and the pendulum mass concentrated in the bob, L is the length of the pendulum.
To simplify the algebraic manipulation in the forthcoming developments, we normalize the above equations by introducing the following scaling transformations,

$$q = x/L,\ u = f/(mg),\ d\tau = dt\sqrt{g/L},\ \delta = M/m$$

This normalization leads to the simpler system,

$$\cos\theta\,\ddot{q} + \ddot{\theta} - \sin\theta = 0$$
$$(1+\delta)\ddot{q} + \cos\theta\ddot{\theta} - \dot{\theta}^2\sin\theta = u,$$

where, with an abuse of notation "." stands for differentiation with respect to the dimensionless time τ. Then, a convenient partial feedback linearization input is proposed as follow (see Spong),

$$u = \cos\theta\sin\theta - \dot{\theta}^2\sin\theta + v\left(\sin^2\theta + \delta\right)$$

which produces the feedback equivalent system:

$$\ddot{\theta} = \sin\theta - \cos\theta v, \qquad (1)$$
$$\ddot{q} = v$$

Notice that for the new input $v = 0$ and $\theta \in [0, 2\pi]$ the aforementioned system has two equilibrium points one is an unstable equilibrium point $(\theta, \dot{\theta}, q, \dot{q}) = (0,0,0,0)$ and the other is a stable equilibrium point $(\theta, \dot{\theta}, q, \dot{q}) = (\pi, 0, 0, 0)$. The issue is to stabilize the system around its unstable equilibrium point, *i.e.*, we wish to bring the pendulum to its upper position and the cart displacement to zero simultaneously.

3 A Practical control law

Traditionally, the problem of designing a stabilizable input control law for system (1) is based on the well known Lyapunov method. Roughly speaking, it consists of proposing a positive definite function (*or Lyapunov function*) provided that, its time derivative along the trajectories of system (1) be at least semi-definite. The very hard problem is how to find *the Lyapunov function*. We relax the problem by setting off a

semi-definite instead of a positive definite function, and then the asymptotic stability is assured by a simple linearization of the closed loop-system.
Let us first introduce the following auxiliary variable:

$$\xi(x) = \sin\theta + k_p\theta + k_d\dot{\theta} + (k_p q + k_d \dot{q})\cos\theta + \alpha\dot{q} \tag{2}$$

Where x is the vector state defined as $x = [\theta, \dot{\theta}, q, \dot{q}]$. The design parameters k_p, k_d and α are positive constants that will be selected later.

Next, let us introduce the following semi-definite function,

$$V(x) = \frac{1}{2}\xi^2(x) \tag{3}$$

The time derivative of $V(.)$ along the trajectories of the system (1) is then given, by

$$\dot{V}(x) = \xi(x)(\Omega(x) + \alpha v)$$

where

$$\Omega(x) = k_d \sin\theta + (k_p + \cos\theta - k_p \sin\theta)\dot{\theta} + (k_p \cos\theta - k_d \sin\theta\dot{\theta})\dot{q} \tag{4}$$

proposing v provided that

$$\Omega(x) + \alpha v = -K_l \xi(x)$$

Clearly, we have.

$$v = -\frac{1}{\alpha}(K_l \xi(x) + \Omega(x)) \tag{5}$$

Comment: The differential equations set formed when by system (1) and the feedback force v given in (5), is referred to as the non-linear closed-loop system.

Surprisingly, the proposed control law v turned out to be able to asymptotically stabilize the system (1) around the unstable equilibrium point $x = 0$, for a large attraction domain[2]. $D \subset R^4$. Where the control parameters $\{K_i, Kp, K_d, \alpha\}$ are selected such that the linearization of the closed-loop system around of $x = 0$ is Hurwitz, as we will show in the following proposition, which summarizes the previous discussion as:

Proposition: The nonlinear closed-loop system is locally asymptotically exponentially stable to a desired equilibrium point $x = 0$ if the design parameters K_i, k_p, k_d and α are chosen so that the following polinomial in the complex variable s *is Hurwitz*:

$$p(s) = s^4 + \left(Ki - \frac{1}{\alpha}\right)s^3 - \left(1 + \frac{k_d}{\alpha} + \frac{K_i}{\alpha}\right)s^2 - \left(Ki + \frac{kdKi}{\alpha} + \frac{k_p}{\alpha}\right)s - \frac{K_i K_p}{\alpha} \tag{6}$$

Proof:

The Jacobian linearization of the closed-loop system (see (1) and (5)) around the desired equilibrium point $x = 0$, is given by: $\dot{x} = Ax$, where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 + \frac{k_d + K_i(1 + k_p)}{\alpha} & \frac{1 + k_d K_i + K_p}{\alpha} & \frac{K_i K_p}{\alpha} & \frac{(\alpha + K_d)K_i - k_p}{\alpha} \\ 0 & 0 & 0 & 1 \\ -\frac{k_d + K_i(1 + K_p)}{\alpha} & \frac{-1 - K_d K_i - K_p}{\alpha} & -\frac{K_i k_p}{\alpha} & \frac{-(\alpha + k_d)K_i - k_p}{\alpha} \end{bmatrix},$$

computing the characteristic polynomial of matrix A, we have the expression (6). Hence, selecting the design parameters such that $p(s)$ is Hurwitz, we guarantee that the closed-loop system be at least locally asymptotically exponentially stable.

Note that if the parameters K_i k_p and k_d are positive, it is necessary that α be negative

Remark: Eventually, we will try to rigorously make the estimation of the attraction domain D by means of Lyapunov functions in conjunction with La Salles's Theorem. It is worth mentioning that an efficient estimation of the set D is a difficult problem. It turns out to be a very demanding process in terms of computational resources, because it involves an optimization problem. In other cases, it is necessary to solve a partial

[2] where $D = x(0) \in R^4 / \lim_{t \to \infty} x(t) = 0$

differential equation (see [13] and the references included in this paper). For this reason, we believe that the detection of D is beyond the scope of this work.

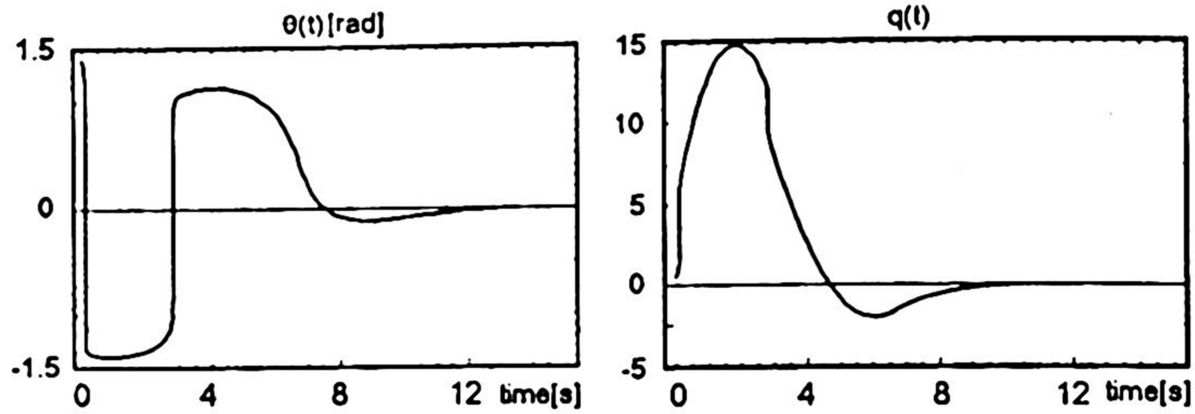


Fig. 2. Parameter α

4 Numerical Simulations

Simulations were carried out to evaluate the efficiency of the proposed feedback controller for three experiments. The experiments were implemented in *Mathematica* by means of a traditionally Runge-Kutta algorithm; the step size of the method was chosen to be equal to 0.001.

In the first experiment, we used the proposed controller (5), when it was applied to the nonlinear model (1). The design parameters were as $K_i = 10, k_p = 0.499, k_d = 1.272$ and $\alpha = -0.1$ and the initial conditions were set as $\theta(0) = 1.45, \dot{\theta}(0) = 0.2, q(0) = 0.5$ and $\dot{q}(0) = 0$. Figures 2 and 3 show the closed loop responses of the feedback equivalent system.

In the second experiment, we considered the design parameters and the initial conditions as before, but introduced a dissipative force in the unactuated direction, i.e. we add the damping $-\gamma\dot{\theta}$ into the first differential equation of the model (1), with $\gamma = 0.3$ Figures 4 and 5 show the robustness of the proposed nonlinear control when damping is considered in the numerical simulations. Notice that it is not generally true that damping makes a feedback-stabilized equilibrium asymptotically stable. That is to say, damping in the unactuated direction $\theta - direction$ tends to enhance stability while damping in the actuacted direction $y - direction$ tends to destabilize (see [14] and [15]).

Finally, we considered a swinging task to bring the pendulum from the lower resting position $x(0) = (\pi, 0, 0, 0)$ to the upper resting position $x(t) = 0$. To accomplish it, we first used the input v to produce an instability in order to pull over the pendulum from the downward angular position. Then, when the pendulum was on the upper angular position, input v was switched to make that unstable equilibrium point became a asymptotically stable point. The switch in v was carried out by changing parameter α this was done taking $\alpha = -0.1$ when $\theta \in [-\pi\ 2, \pi\ 2]$ and $\alpha = -0.1$ in other cases. The other design parameters were set as before.

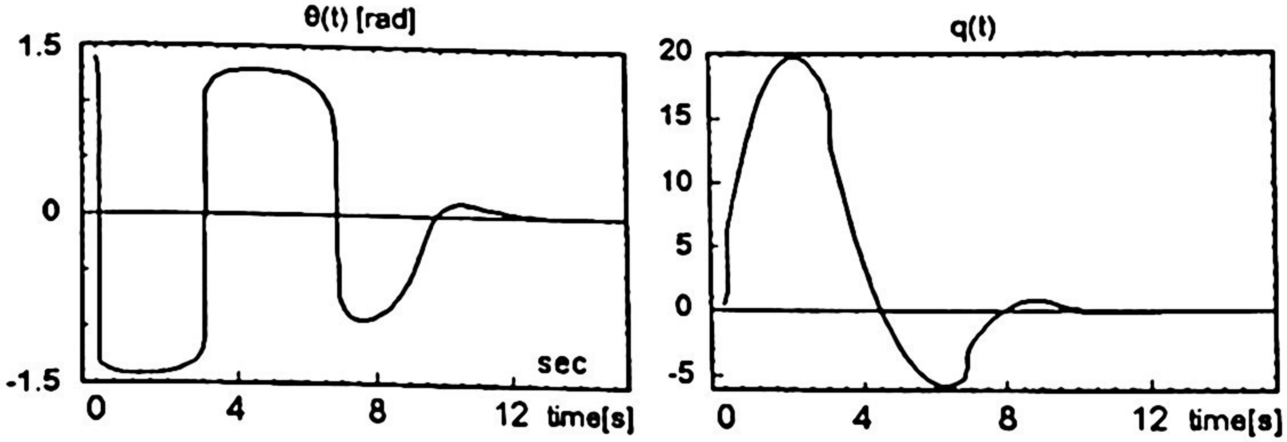


Fig. 3. Closed-loop responces when damping is considered α

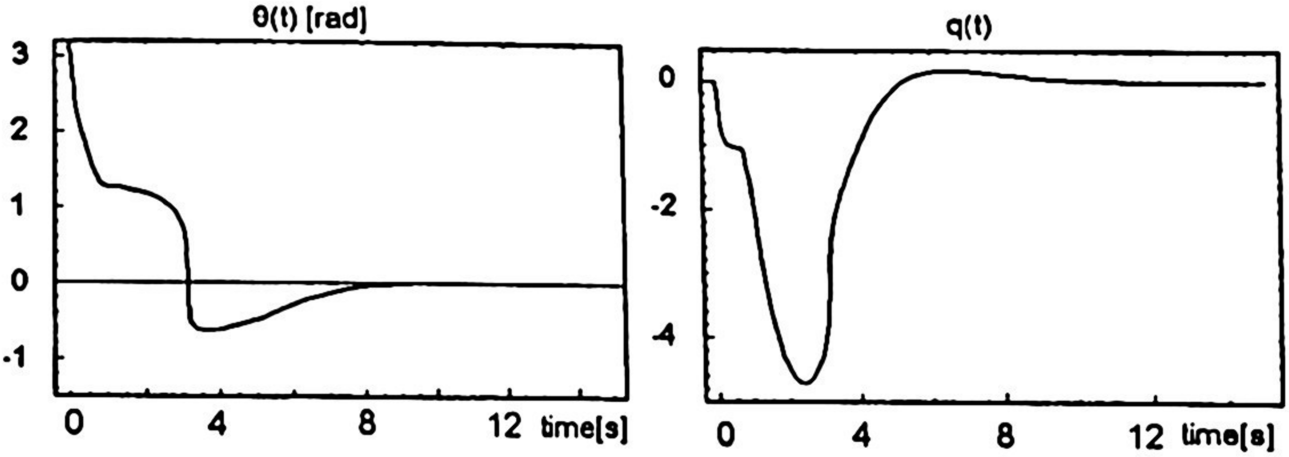
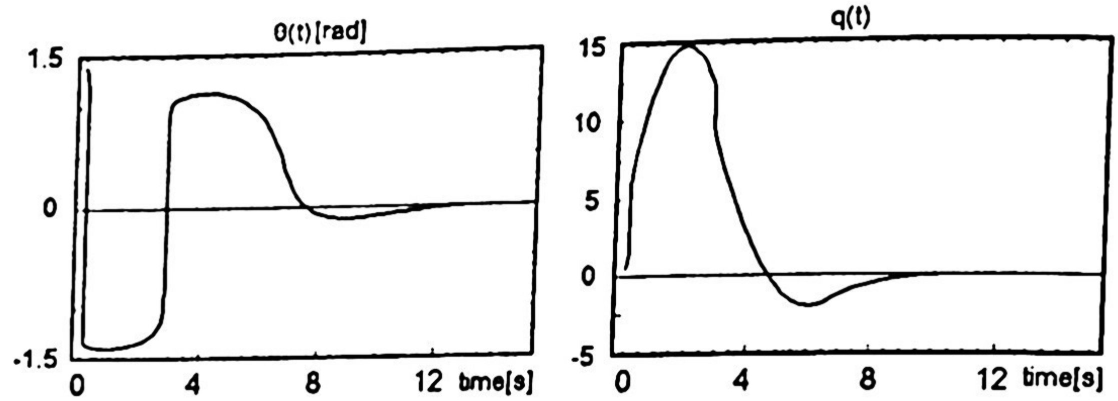


Fig. 4. The transfer from lower resting to upper resting positions

5 Conclusions

We have presented a practical nonlinear control for the inverted pendulum cart system. The goal of the proposed controller is to change the unstable equilibrium point into a locally asymptotically stable equilibrium point. The advantages of the presented controller is that it has a very large attraction domain, as we showed in the numerical simulations. It is robust with respect to dissipative forces. Also, the controller allows us to establish a simple strategy to bring the pendulum from the downward resting position towards the upward resting position, by means of a simple change of sign of differential equation (see [13] and the references included in this paper). For this reason, we believe that the detection of D is beyond the scope of this work.



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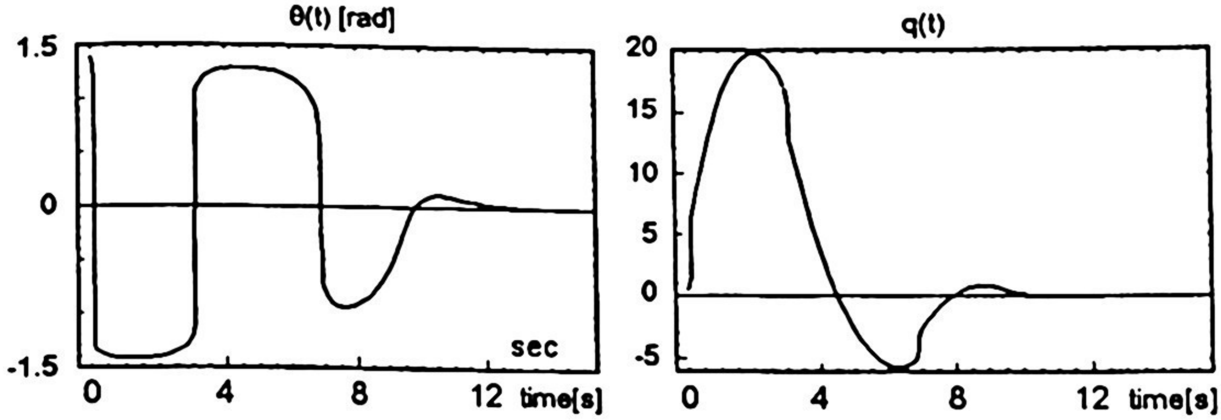

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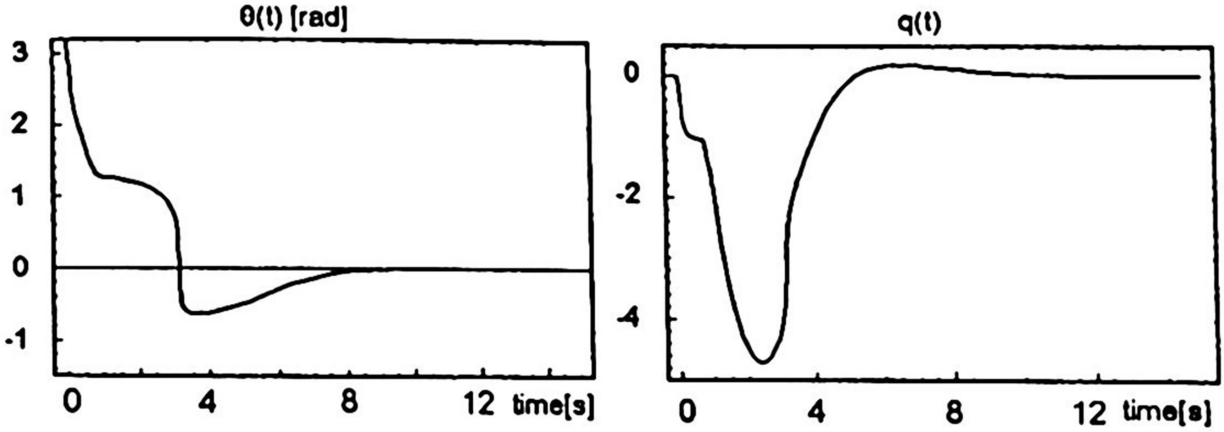

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